Chapter 0

Introduction

This initial chapter contains a brief introduction to the theory of the Riemann zeta-function that has provoked several unanswerable questions. In the book proposed for your attention, the author suggests an innovative approach to the subject. While traditional models are still widely used, it is to be hoped that our new vision will initiate a discussion which may offer a final solution to the Riemann Hypothesis — the most important problem of the modern Mathematics.

0.0 Zeta-function

0.0.0 Definition

Before we start out discussing the Riemann Hypothesis on the zeros of $\zeta(s)$, we need some more background. Our concern here is the precise definition of the main object for the further study. So, let us state n^{-s} as $e^{-s\ln n}$, where $\ln n$ is a natural logarithm. It should be fixed that from now on we denote by ϵ an arbitrarily small real positive number.

Quite naturally, the Riemann *zeta-function* is now defined as the Dirichlet sum

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s = \sigma + it,$$

where the series converges absolutely and uniformly in each half-plane $\sigma > 1 + \epsilon$. The importance of this function grows from the fact that the sum of the series may possess a holomorphic continuation outside its region of convergence.

Before the discussion on this essential statement, we now introduce some new notation which seems convenient for the present book.

0.0.1 Analytic continuation

Because of the importance which has been placed upon the analytic character of $\zeta(s)$, we show first that the zeta-function admits an analytic continuation at least into the *critical strip*

$$0 < \sigma < 1.$$

We prefer to start with the most simple construction as possible, although it is interesting to keep in mind different ways of analytic continuation as providing steps towards more general ζ- and L-functions. For this purpose, let us consider the Dirichlet *eta-function* defined via the series

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}, \quad \sigma > 0,$$

which is uniformly but not absolutely convergent on compacts in the right half-plane. One can easily note that

$$\left(1 - \frac{2}{2^s}\right) \zeta(s) = \eta(s), \quad \zeta(s) = \left(1 - \frac{2}{2^s}\right)^{-1} \eta(s),$$

and we obtain an analytic continuation of the zeta-function into the critical strip as desired.

0.0.2 Xi-function

In fact, the Riemann function $\zeta(s)$ has analytic continuation over the whole complex plane with a simple pole at $s = 1$ as its only singularity. Several standard properties of this extension will be stated here, while the rigorous proofs are readily found in the literature.

Before looking at this matter, however, it is worth considering a very important entire function of the first order

$$\xi(s) = \frac{s(s-1)}{2} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s), \quad s \in \mathbb{C},$$

namely the Riemann *xi-function.* In order that our description should be as complete and clear as possible, we will give an account of its properties in the next paragraphs.

0.0.3 Functional equation

In reading this chapter, a slight familiarity with the theory of entire functions might stimulate the deeper understanding of the whole picture. It is to some extent possible to say, with hindsight, that the theory of the zeta-function was the cradle of the theory of entire functions. For instance, the Weierstrass product represents one of the fundamental results and was invented to justify the expression

$$\xi(s) = \xi(0) \prod_{\varrho} \left(1 - \frac{s}{\varrho}\right), \quad s \in \mathbb{C}$$

that is known as the Hadamard decomposition over the zeros.

Notice that the choice of the definition for the xi-function was not accidental; it is determined by deep reasons. It can be easily seen from the *functional equation*

$$\xi(s) = \xi(1-s)$$

that the Riemann zeta has its *trivial zeros* at the points $s = -2, -4, -6, \ldots, -2n, \ldots$, while the so-called *nontrivial zeros* ϱ are located in the critical strip.

0.0.4 Euler product

The zeta - function itself with its basic properties, including the functional equation for the real values of the argument s, was originally discovered by Euler [75]. Modern mathematicians are indebted to Riemann for extending of this definition to the complex domain. His complex zeta - function can also be decomposed into the convergent *Euler product*

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \sigma > 1,$$

where p runs through all primes. Nontrivial zeros are located in the critical strip due to this decomposition, and there is no zeta zeros in the convergence domain. One can easily note that the inverse of the given product

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \quad \sigma > 1$$

is closely connected with the famous *Möbius function* with some interesting implications in relation to questions of asymptotic enumeration.

Although Riemann wrote only one paper [204] on the subject under consideration, that one was really groundbreaking. He justified the analytic continuation of $\zeta(s)$ over the complex plane $s \in \mathbb{C}$ and found the *explicit formula*

$$J(x) = \mathrm{Li}(x) - \sum_{\varrho} \mathrm{Li}\left(x^{\varrho}\right) - \ln 2 + \int_x^{\infty} \frac{dt}{t(t^2-1)\ln t}$$

provided that the notation key

$$\mathrm{Li}(x) = \int_2^x \frac{du}{\ln u}$$

is used for the integral logarithm.

According to it, the distribution of zeta zeros has influence on the distribution of prime numbers. The interplay between the Hadamard decomposition and the Euler product leads to the well - known formula for the prime counting function

$$\pi(x) = \#\{p : p \le x\} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} J\left(x^{1/n}\right).$$

0.0.5 Riemann Hypothesis

He also conjectured that all nontrivial zeros of the zeta - function lie on the *critical line* $\sigma = 1/2$. The statement of the conjecture can be formulated in the slightly different way.

Riemann Hypothesis. *All nontrivial zeta zeros ϱ lie on the critical line. In other words, all zeros of the entire function*

$$\Xi(z) = \xi\left(\frac{1}{2} + iz\right)$$

should be real numbers.

This conjecture drew much attention; if it is true, we get the best possible estimate in the *prime number law*

$$\pi(x) = \mathrm{Li}(x) + O\left(x^{1/2+\epsilon}\right), \quad x \to \infty.$$

It seems unknown how Riemann was led to his Hypothesis, but he knew much more about the subject than he cared to publish.

0.0.6 Probabilistic argument

From now on, we will abbreviate the Riemann Hypothesis

$$\mathrm{Re}\,\varrho = \frac{1}{2}, \quad \zeta(s) \neq 0, \quad \sigma > \frac{1}{2}$$

to the shortened form RH. According to Littlewood [37], RH is valid if and only if the series

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

converges to the right of the critical line $\sigma > 1/2$.

Basing upon this remark, Good and Churchhouse demonstrated [95] that RH seemed rather natural from the probabilistic standpoint; application of the Abel transform shows that the *Mertens function* must behave as a trajectory of the Brownian motion

$$M(x) = \sum_{n=1}^{[x]} \mu(n) = O\left(x^{1/2+\epsilon}\right), \quad x \to \infty.$$

Biane, Pitman and Yor noted [20] that the values of $\xi(s)$ might be seen as the generalized moments of the Brownian bridge.

0.0.7 Asymptotics of zeros

The asymptotics of zeros in the critical strip $0 < \sigma < 1$ comes from the formula which was proved by von Mangoldt; the asymptotics of zeros on the critical line comes from another formula which was rediscovered by Siegel. Riemann considered it very likely that the roots of $\Xi(z)$ should lie on the real line, but he was not able to prove it. He put all attempts aside, since it was not necessary for his main goal — the formula for the number of primes less than the given magnitude.

0.0.8 Elementary results

RH seems not to be so terribly difficult at the first glance: for example, all zeros of the function

$$\Xi(z + ia) + \Xi(z - ia), \quad a > 0$$

must lie on the real line. Unfortunately, this leutnant Taylor theorem does not work in the most interesting case $a = 0$.

We are not going here to touch upon with the various intriguing equivalent forms of RH due to Grommer, Riesz, Li, etc., etc. Most of them look plausible, but nobody was able to apply them up to this day. The abandoned Jensen - Pólya approach was revived, however, in the recent work of Griffin, Ono, Rolen and Zagier [102].

0.0.9 Characteristic function

Anyhow, the doubt always remains as to whether we have indeed found all the roots. Note, to begin with, that the Riemann Xi-function

$$\Xi(z) = 4\int_0^\infty \Phi(u)\cos(zu)\,du$$

may be considered as a characteristic function of the unimodal probability density

$$\Phi(u) = \sum_{n=1}^{\infty}\left(2n^4\pi^2 e^{9u/2} - 3n^2\pi e^{5u/2}\right)e^{-n^2\pi e^{2u}},$$

which is closely connected with the third Jacobi theta-function. By using analytic properties of $\Phi(u)$ and the theory of moments, it is possible to prove the first nontrivial fact: *there are infinitely many zeta zeros along the critical line.* This theorem was originally proved by Hardy [108] in a more complicated way.

0.1 Zero-free region

0.1.0 Trigonometric sums

The deep problem of the zeta zeros localization needs much more profound and effective methods. Let us consider the Dirichlet series from **0.0.0** and denote by

$$\zeta_N(s) = \sum_{n=1}^{N}\frac{1}{n^s}$$

the correspondent truncated sum. Turán discovered a remarkable connection between RH and these partial zeta sums. It is not without interest to note that RH should be valid if there exists the number N_0 such that

$$\zeta_N(s) \neq 0, \quad \sigma > 1, \quad N > N_0.$$

But unfortunately, there is no N_0 at all; Voronin and Montgomery showed that Turán's remark could not be used to prove RH [176].

0.1.1 Approximate functional equation

By using standard properties of the gamma-function, one can rewrite the functional equation from **0.0.3** as

$$\zeta(s) = \chi(s)\zeta(1-s), \quad \chi(s) = 2^s\pi^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right).$$

At this stage, it may be interesting to see what information it yields when applied to the truncated zeta sums

$$\zeta_x(s) = \sum_{n\le x}\frac{1}{n^s},$$

which appear as a useful tool in different topics of the zeta-function theory.

Perhaps the best form of all approximate functional equations is the following classical result due to Hardy and Littlewood [111]: *uniformly in* σ *we have*

$$\zeta(s) = \zeta_x(s) + \chi(s)\zeta_y(s-1) + O\left(x^{-\sigma}\right) + O\left(t^{1/2-\sigma}y^{\sigma-1}\right),$$

supposing that

$$0 \le \sigma \le 1, \quad x, y, t > C > 0, \quad 2\pi xy = t.$$

It is not obvious that such analysis should lead to results of great theoretical importance, but let us point out that the Riemann - Siegel formula, briefly discussed in **0.0.7**, may be seen as an approximate functional equation on the critical line.

0.1.2 Hardy function

It is beyond the scope of this book to develop these matters further than is needed for a preliminary understanding, but it may be illuminating to note that

$$\left|\chi\left(\frac{1}{2}+it\right)\right| = 1, \quad \chi\left(\frac{1}{2}+it\right) = e^{-2i\vartheta(t)},$$

where we write

$$\vartheta(t) = -\frac{1}{2}\arg\chi\left(\frac{1}{2}+it\right).$$

Hardy used the properties of the function

$$Z(t) = e^{i\vartheta(t)}\zeta\left(\frac{1}{2}+it\right)$$

in his original proof of the result from **0.0.9** that was already mentioned above.

This *Hardy function* can be considered in our current conception of the approximate functional equation. The error term is represented by an integral in the complex domain whose evaluation leads to an asymptotic expansion

$$Z(t) = 2\sum_{n=1}^{N}\frac{\cos\{\vartheta(t) - t\ln n\}}{\sqrt{n}} + (-1)^{N-1}\left(\frac{2\pi}{t}\right)^{1/4}\frac{\cos\{t-(2N+1)\sqrt{2\pi t}-\pi/8\}}{\cos(\sqrt{2\pi t}\,)} +$$

$$+ O\left(t^{-3/4}\right), \quad t \to \infty.$$

For the expert on analytic number theory, we must emphasize that the present book outlines only the general scope of ideas, making no attempt at deducing the sharpest estimates in each case.

0.1.3 Logarithmic derivative

The logarithmic differentiation of the Hadamard decomposition gives the formula

$$\frac{\zeta'(s)}{\zeta(s)} = B - \frac{1}{s-1} + \frac{1}{2}\ln\pi - \frac{1}{2}\psi\left(\frac{s}{2}+1\right) + \sum_{\varrho}\left(\frac{1}{s-\varrho}+\frac{1}{\varrho}\right),$$

where $\psi(s)$ stands for the logarithmic derivative of the gamma - function; applying the argument principle, we can find the precise expression for the number of zeta zeros in the region $0 < \sigma < 1,\ 0 < t < T$. As it was said in **0.0.7**, we came to the *Riemann - von Mangoldt formula*

$$N(T) = \frac{T}{2\pi} \ln \frac{T}{2\pi} - \frac{T}{2\pi} + O(\ln T), \qquad T \to \infty.$$

On the other hand, logarithmic differentiation recasts the Euler product into another, more convenient form

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s},$$

where the coefficients of the series are known as the values of the arithmetic von Mangoldt function. Their nonnegativity, combined with the famous trigonometric equality

$$3 + 4\cos(z) + \cos(2z) = 2\{1 + \cos(z)\}^2,$$

implies that

$$\zeta^3(\sigma)\, |\zeta(\sigma + it)|^4\, |\zeta(\sigma + 2it)| \geq 1$$

for all $\sigma > 1$. Immediately, we obtain the classical result of Hadamard and de la Vallée - Poussin: *there is no zeta zeros on the border of the critical strip*

$$\zeta(1 + it) \neq 0, \qquad -\infty < t < \infty.$$

If we put von Mangoldt's ideas in this context, we can establish the zero - free region

$$\zeta(\sigma + it) \neq 0, \quad \sigma > 1 - C/\ln t, \quad t > 2,$$

which is only of historical interest now, since better regions are known at present.

0.1.4 Better results

The problem of the zero - free region is of great importance from a theoretical point of view. Although all attempts to prove the *weak Riemann Hypothesis*

$$\zeta(s) \neq 0, \quad \sigma > \sigma_0, \quad \frac{1}{2} < \sigma_0 < 1$$

have failed so far, it was possible to improve the zero - free domain substantially, and one of the sharpest region was given in the form

$$\sigma > 1 - \frac{C}{\ln^{2/3} t\, \ln^{1/3} \ln t}, \quad t > t_0.$$

It turns out that the quality of the result depends on the region where the estimate of growth

$$\zeta(s) = O\left(\ln^A t\right), \quad t \to \infty$$

holds for some positive power A. As soon as $\zeta(1+it) = O(\ln t)$, one should estimate the order of zeta near the borderline $\sigma = 1$ in the course of the proof. It is seen through the approximate functional equation that the task depends on the estimation of the so - called trigonometric sums.

0.1.5 Estimates

Estimates for these trigonometric sums arise as an application of the famous method that was given by Vinogradov [241] and Korobov [147]. Their way of estimating sums via estimating trigonometric integrals supersedes the earlier approaches by Weyl [247] and van der Corput [53]. It has something in common with the older circle method [110] of Hardy and Littlewood and is of great use in other branches of analytic number theory.

0.1.6 Possible exceptions

Many experts believe that RH should be true because possible exceptions are rare. In the spirit of **0.1.3**, let us consider the number $N(\sigma_0, T)$ of zeta zeros in the region $\sigma_0 < \sigma < 1,\ 0 < t < T$. It will be the natural density measure for the distribution of zeta zeros off the critical line. For example, the simplest estimate is given by the following statement:

$$N(\sigma_0, T) = O\left(T^{4\sigma_0\{1-\sigma_0\}} \ln^{12} T\right), \quad T \to \infty,$$

which holds for all

$$\frac{1}{2} < \sigma_0 < 1.$$

The results of this type provide information about the distribution of primes in short intervals; one can compare them with the Riemann - von Mangoldt formula that was given in **0.1.3**.

0.1.7 Counter - examples

Although sometimes the role of the Euler product is a little bit overestimated, we ought to mention that it is essential in the subject. The Davenport - Heilbronn function shares all basic properties of the Riemann zeta - function: it can be represented by the Dirichlet series, it has the functional equation and so on; but it has no Euler product, and the analogue of RH is not true. It is well - known that some of its zeros are off the critical line, and it is the same for the Epstein zeta - function.

0.1.8 Zeta argument

Collaterally, our understanding of RH will be better after the full study of the function

$$S(T) = \frac{1}{\pi} \arg \zeta \left(\frac{1}{2} + iT\right).$$

Indeed, it is not so difficult to verify that

$$N(T) = \frac{1}{2\pi} T \ln T - \frac{1 + \ln 2\pi}{2\pi} T - \frac{7}{8} + S(T) - O\left(\frac{1}{T}\right), \quad T \to \infty,$$

but the theory of $S(T)$ is one of the most complicated parts of the whole subject. The recent works by Boyarinov [32] and Korolev [148] contain deep achievements in this direction.

0.1.9 Sieves

It is necessary to underline that almost every object in the zeta-function's theory has its counterpart in arithmetics, especially in the general theory of sieves. RH is quite often unnecessary and can be avoided or eliminated with the help of the large sieve.

0.2 Critical strip

0.2.0 Lindelöf Hypothesis

The *Lindelöf Hypothesis* (LH) which says that

$$\zeta\left(\frac{1}{2}+it\right)=O(t^{\epsilon}), \quad t\to\infty$$

is weaker than RH that implies not only LH but also the estimate

$$\frac{1}{\zeta(s)}=O(t^{\epsilon}), \quad \sigma>\frac{1}{2}.$$

Indeed, if RH is true, then $\ln\zeta(s)$ is regular for $\sigma>1/2,\ s\neq 1$. The estimates above are due to Littlewood [164] and follow from the Borel-Carathéodory lemma and Hadamard's three-circles theorem. Unless one is willing to accept RH, our state of knowledge is not satisfactory, and the problem of determining the precise order of zeta in the critical strip is yet unsolved.

0.2.1 Mean values

The easier problem of an average order (*mean value*) has been solved at least in the most simple cases. Estimates of the moments

$$I_k(T)=\int_0^T\left|\zeta\left(\frac{1}{2}+it\right)\right|^k dt$$

play an outstanding role in the theory of $\zeta(s)$. However, all upper estimates were insufficient to prove LH, while the lower ones were insufficient to disprove it.

0.2.2 Density of exceptions

The problem of the order in the critical strip may be stated in terms of the constant

$$\limsup_{t\to\infty}\frac{\ln|\zeta(\sigma+it)|}{\ln t}.$$

According to LH, this constant should be equal to zero at $\sigma=1/2$. For LH to be true, it is necessary and sufficient that the density estimate

$$N(\sigma,T+1)-N(\sigma,T)=o(\ln T), \quad T\to\infty$$

holds for all $\sigma>1/2$. It should be also pointed out that this is true under RH, but unconditionally we know much weaker results.