**On the Riemann Hypothesis**

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**Abstract**

The author proposes an elementary quantum field theory with a mass gap: all absolute values of the Riemann zeta-function's derivative at zeta zeros are uniformly bounded from below by a positive constant. This theorem speaks in disfavour of the Riemann Hypothesis.

**1. Main result.**

The Riemann zeta-function is defined as the Dirichlet sum

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*

where the series converges absolutely.

In fact, it has an analytic continuation over the whole complex plane with a simple pole \*\*\*\*\* as its only singularity.

It can be easily seen that \*\*\*\*\* has its trivial zeros at the points

\*\*\*\*\*\*\*\*\*, while the nontrivial zeros \*\*\*\* are located in the strip \*\*\*\*\*

and supposedly lie on the so-called critical line

\*\*\*\*\*\*\*\*.

Under this Riemann Hypothesis, Ng conditionally proved that

\*\*\*\*\*\*\*\*\*\*\*\*.

The author of the present paper suggests an innovative approach to the subject and proves the following unconditional statement.

Theorem. *The Riemann Hypothesis is false because* \*\*\*\*\*.

We consider the values \*\*\*\*\*\*\* as the energy levels of elementary excitations in a finely chosen oscillating system.

It is to be hoped that our new vision will initiate a discussion which may offer a final solution to the most important problem of the modern Mathematics.

**2. Positive definiteness.**

Before we start out discussing the Riemann Hypothesis, we need some more background. Our concern here is the precise definition of the positive definite functions for the further study.

Definition. A symmetric function

\*\*\*\*\*\*\*\*\*\* is called *positive definite* in its unbounded domain of existence

\*\*\*\*\*\*\*\*\* if and only if the following condition

 \*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\* for all finite sets of coefficients and arguments

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*.

For bounded domains this notion is defined in the same manner.

In order to construct new functions, we will apply the laws listed below:

% it can be easily derived that both kernels

\*\*\*\*\*\*\*\*\*\*\* should also be positive definite for each \*\*\*\* and \*\*\*\*\*\*\*\*;

% in particular, functions of the *product type*

\*\*\*\*\*\*\*\*\*\*\*\* are always positive definite;

% multiplication by the positive number preserves positive definiteness;

% given the sequence of positive definite kernels

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*, one can see that the sum of the series

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\* must be positive definite in its domain of convergence;

% moreover, an integral function \*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\* should be a positive definite kernel provided

that the slices \*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\* enjoy the same property.

From now on, we will use these rules without special announcement.

**3. Reproducing kernels.**

Let us start with the following statement.

Proposition. *Suppose that* \*\*\*\*\*\*\*\*\*\*\*\*\*. *Then the kernel of two variables*

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\* *is positive definite.*

Proof. First of all, let us put \*\*\*\*\*\*\*. Due to the binomial formula

\*\*\*\*\*\*\*\*\*\*\*\*\*, we get a decomposition

\*\*\*\*\*\*\*\*\*\*\*\*\*\* that converges on the bounded interval \*\*\*\*\*\*\*\*\*.

One can easily see that the kernel

\*\*\*\*\*\*\*\*\*\*\*\*\* is a sum of a series of the positive definite product functions

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\* with nonnegative coefficients.

Thus, it is a positive definite function. The situation does not change after multiplying by the product function, so the kernel \*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\* is still positive definite.

For the same reasons, another function

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\* turns out to be positive definite.

In fact, it appears possible to write

 \*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*, and we also have

 \*\*\*\*\*\*\*\*\*\*\*\*\* in this setting.

 Termwise subtraction gives us the Taylor series

\*\*\*\*\*\*\*\*\*\*\*\*\*\*

with nonnegative coefficients, which are subject to the prerequisite condition

\*\*\*\*\*\*\*\*\*\*\*. By convention, the record

\*\*\*\*\*\*\*\*\*\*\*\* is equal to \*\*\* for \*\*\*\*\*\*.

The doubled function, in its turn, is also positive definite.

By adding it to the previous one, we obtain the new positive definite kernel

\*\*\*\*\*\*\*\*\*\*\*\*\*\*.

It remains to add the positive definite function

\*\*\*\*\*\*\*\*\*\*\*\*\*.

Here the coefficient

\*\*\*\*\*\*\*\* is positive because \*\*\*\*\*\*, and the sum

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\* is all the more positive definite.

Without loss of generality, we can change \*\*\*\*\*\* to \*\*\*\*\*\* while preserving the range of values.

The following statement can be deduced from the exposition that has already been given.

Proposition. *Fix the value of the parameter*

\*\*\*\*\*\*\*.

Then the function \*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\* is positive definite.

Proof. Let us produce a suitable change of variables

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\* that keeps the positive definiteness of the function

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\* from the previous item \*\*\*\*\*\*\*\*. Look at the calculations below.

Firstly, we get the equality

\*\*\*\*\*\*\*\*\*\*\*\*\* and secondly, the identity

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*. Plug these outcomes in the above mentioned form.

As a result, we obtain the positive definite function

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*.

Furthermore, the product function before the square brackets makes no influence on positive definiteness.

So does the division by \*\*\*\*\*.

We need the following auxiliary assertion.

Proposition. *The function*

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*

is positive definite under the same conditions.

 Proof. Basing on the formula for the Gamma distribution, we can write

\*\*\*\*\*\*\*\*\*\*\*\*\*\*. The similar representation of the shifted kernel

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\* also takes place.

The difference

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\* has the required property.

Indeed, this integral is a continuous sum

of the positive definite functions

\*\*\*\*\*\*\*\*\*\*\*\*\*\*

of the product type. These functions were taken with the positive weight

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*.

Finally, the division by the product function with the positive external coefficient

\*\*\*\*\*\*\* do not change the positive definiteness of the whole expression.

We came to the following conclusion.

Proposition. *The function* \*\*\*\*\*\*\*\*\*\*\*\*

is positive definite; here all parameters change within their former limits.

Proof. One can add the function

\*\*\*\*\*\*\*\* from the item \*\*\*\*\*\*\*\*\*

 to the function

\*\*\*\*\*\*\*\*\*\*\*\*\* from the item \*\*\*\*\*\*\*. Both of them are positive definite, and their sum is as well.

Another observation concerns the double integral. Besides, it can be considered as a correlation function of some integrated stochastic process, so its positive definiteness arises from

the Aronszajn-Kolmogorov theorem.

Proposition. *The function*

\*\*\*\*\*\*\*\*\*\*\*\*\* *is positive definite in the same circumstances.*

Proof. Take into account the coefficient \*\*\*\*\*\*\*\*\*\*\*.

The expression in the square brackets has the form of the positive definite function from the item \*\*\*\*\*\*\*. Due to Merser's theorem, the integrand can be decomposed into an uniformly convergent series of product functions.

 Double integration leads to another series of this type, and its sum is obviously positive definite.

The key outcome of this section is a proof of the following statement.

Proposition. *As usual, we get* \*\*\*\*\*\*\*\*\*\*. *The function of two variables*

\*\*\*\*\*\*\*\*\*\*\*\*\*\* *is positive definite.*

Proof. Refine the double integral from the item \*\*\*\*\*\*\*

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*.

The multiplier before the square brackets does not affect the final conclusion.

**4. Coupling constant.**

Now, before we go any further, let us establish a fact of great importance. The rest of the paper will throw light on its close links with Coulomb's, or better to say, Yukawa's potential.

Proposition. *There exists a constant* \*\*\*\* *such that*

*the function*

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*

*is positive definite; the domain of its definition remains the same.*

Proof.

We can even estimate the numerical value of the absolute constant.

Proposition. *One can take* \*\*\*\*\*\*\*\*.

Proof.

It is necessary to mention one important conclusion that follows from the combination of the occurrences which have been observed.

Proposition. *The function of two variables*

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*

 *is positive definite in its existence domain*

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\* *with the above mentioned constant.*

Proof. Recall that the sum of two positive definite functions is positive definite.

All is based on this principle.

It suffices to add together the function

\*\*\*\*\*\*\*\*\*\*\*\* from the item \*\*\*\*

and the function

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*

from the item \*\*\*\*\*\*.

**5. Gibbs distribution.**

Let us turn to the explicit calculation of the double inverse Laplace transform. We prefer to promptly show the answer, and then we shall justify it.

Proposition. *Under the standard conditions*

\*\*\*\*\*\*\*\*\*\*\*\*, *the kernel defined by the formula*

\*\*\*\*\*\*\*\*\*\* *is positive definite with the same constant.*

Proof. It is worth saying that any kernel is positive definite simultaneously with its double Laplace transform. This is not difficult to demonstrate by virtue of Merser's theorem and, conversely, on the grounds of the M\"untz-Sz\'asz theorem.

The double Laplace transform of the present kernel is actually proportional to the positive definite function \*\*\*\*\*\*\*\*\* from the item \*\*\*\*\*\*.

Note that the positive proportionality factor \*\*\*\*\*\* preserves positive definiteness.

So far as the multiplication by \*\*\*\*\* shifts the Laplace transform one unit on both variables \*\* and \*\*, everything is based on the operational relation

\*\*\*\*\*\*\*\*\*

that seems to be quite unobviuos at the first glance.

Nevertheless, one can readily derive it from the more clear equation \*\*\*\*\*\*.

According to the standard rules of the Operational Calculus,

 it is sufficient to install the formula for the joint density

\*\*\*\*\*\*\*\*\*\*\*\*

that is easy to check.

Our oscillating system will be placed in the thermostat, and the parameter $\delta$ should be interpreted as a reciprocal temperature.

**6. Entropic factor.**

We introduce a very convenient notation

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*

 which shortens the record and highlights similarities with the entropy.

Proposition. *The modified function*

\*\*\*\*\*\*\*\*\*\*\*\* *is positive definite under ceteris paribus conditions.*

Proof. The kernel from the item \*\*\*\* is still positive definite after the division by the function \*\*\*\*\*\* of the product type. We rewrite this kernel \*\*\*\*\*\*\* in a slightly different way.

Our entropic factor is positive definite by itself, and we will take an advantage of this.

It is not difficult to note that another function

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*

should be positive definite.

Redouble our first function and add another one

\*\*\*\*\*\*\*\*\*\*\*\*\*.

Through all necessary elementary transformations, we eventually obtain

\*\*\*\*\*\*\*\*\*\*\*\*\* the positive definite kernel.

Since \*\*\*\*\*\*, adding the positive definite function

\*\*\*\*\*\*\*\* leads to the desired conclusion.

**7. Microcanonical ensemble.**

Our next expression resembles the collision term of Boltzmann's kinetic equation (Stosszahlansatz).

Proposition. *Under the usual conditions, the function* \*\*\*\*\*\*\*\*\*\* *is positive definite.*

Proof. We use our favorite trick of multiplying by the product function \*\*\*\*\*\*\* in order to deduce this proposition from the statement of the item \*\*\*\*\*\*\*.

The passage from the Statistical Mechanics to the Quantum Field Theory will be based on the dimensionality reduction, like always.

**8. Lagrangian.**

The above mentioned entropic factor is homogeneous:

\*\*\*\*\*\*\*\*. Here we are dealing with homogeneity of the degree minus one.

Proposition. *Each function from the family*

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\* *is positive definite.*

Proof. The change of variables

\*\*\*\*\*\*\* keeps the positive definiteness of the function from the item

\*\*\*\*\*.

We should recall that positive definiteness was defined in the item \*\*\*\* right down to the positive coefficient.

The positive factor \*\*\* is put outside the brackets, and its role may be neglected.

In view of its importance, the next statement receives a special name.

Lemma. *The integral form with the fixed constant*

\*\*\*\*\*\*\*\*\* *is positive definite in the domain*

\*\*\*\*\* *for each value of the main parameter*

\*\*\*\*\*\*\* *and it is homogeneous of the degree minus two.*

Proof.

It is appropriate to introduce a notation \*\*\*\*\*\*\*\* for the Lagrangian.

Its homogeneity is based on the simple change of variables:

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*. This gauge invariance reduces one degree of freedom.

Homogeneity of the entropic factor was already mentioned.

As for positive definiteness, the integral sum of the positive definite slices from the item \*\*\* has the same property.

The common entropic factor can be removed outside the integral sign.

We are going to consider a new positive definite Toeplitz form. It depends only on the difference of the variables.

Corollary. *We get the positive definite function*

\*\*\*\* generated by its characteristic

\*\*\*\*\*\*\*\*\*\*\*\*

when imposing the intrinsic constraint

\*\*\*\*\*\*\* between two previously independent arguments.

Proof. All is based on the main lemma from the item \*\*\*\*. It is not difficult to check the equality

\*\*\*\*\*\*\*\*\* from which positive definiteness easily follows. Owing to homogeneity, we can write

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*.

It suffices to rearrange the variables

\*\*\*\*\*\*\*\*\*\* and multiply by the product of the exponents.

**9. Normal modes.**

We bring our system to normal coordinates that behave just like the gas of uncoupled oscillators.

Proposition. *The characteristic of the Lagrangian*

\*\*\*\*\*\*\*\*\*\* can be decomposed into the double series.

Proof. The statement under consideration is equivalent to

\*\*\*\*\*\*\*\*\*.

In other words, the Lagrangian takes the form

\*\*\*\*\*\*\*\*\*\*\*\*\*.

In order to derive this from its very definition

\*\*\*\*\*\*\*\*\* let us consider the decomposition

\*\*\*\*\*\*\*\*\*\*\*

and integrate it. One can justify such termwise integration, relying on Lebesgue's dominated convergence theorem.

Up to this point we dealt with elementary functions, but it is time to switch to the Riemann zeta.

This will be accomplished by the use of the Fourier transform.

Proposition. We get the formula

\*\*\*\*\*\*\*\* for the spectral density.

Proof. The translation principle of the Harmonic Analysis is assumed to be known. According to this rule, the shifts of \*\*\*\*\*\*\* to all sorts and kinds of possible lags \*\*\*\*\*\* combined with the parallel divisions by \*\*\*\*\*\* are represented as the multiplications by the harmonics \*\*\*\*\*\*\*\* on the level of correspondent Fourier tranforms. Apply the rule to the statement from the item \*\*\*\*\*\*\*.

Here is the first reference to the Riemann zeta. Because of the importance which has been placed upon the values of $\zeta(s)$ inside the critical strip, we show first that the zeta-function admits an analytic continuation at least into this area.

For such purpose, let us consider the Dirichlet eta-function defined via the series

\*\*\*\*\*\*\* that converges uniformly (but not absolutely) on compacts.

 Although it is interesting to keep in mind different ways of analytic continuation as providing steps towards more general functions, we prefer to start with the most simple construction as possible.

Proposition. *We have expressed the spectral density*

\*\*\*\*\*\*\*\*\*\* *through the Riemann zeta-function.*

Proof. The function \*\*\*\*\*\*\*\* from the item \*\*\*\*\*\* is a table integral \*\*\*\*\*.

On the other hand, the equality

\*\*\*\*\*\*\*\*\* holds.

The described method is attractive, due to the fact that the appearance of $\zeta(s)$ is quite logical.

Similar calculations were carried out for the Mellin transform.

Proposition. *The formula* \*\*\*\*\*\*\* *gives the characteristic of the entropic factor.*

Proof. We get it from the definition \*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\* by the direct substitution.

The calculation of the optical transfer function is quite simple.

Proposition. The formula \*\*\*\*\*\*\*\* defines the spectral selection filter.

Proof.

Guided by the statement from the previous item \*\*\*\*\*\*\*, one can note that it is a question of the Fourier transform

\*\*\*\*\*\*\*\*\*\*.

We start from the table integral \*\*\*\*\*\*\*\* and get

\*\*\*\*\*\*. The sum of two integrals gives us the desired answer, resulting in description of the movement of a brownian oscillator with the viscosity \*\*\*\*\*\*.

**10. Green function.**

So when we come to the K\"all{\' e}n\,-\,Lehmann spectral representation, let us recall the statement from the item \*\*\*\*\* and consider two characteristics from the items \*\*\*\* and \*\*\*\*\*.

Proposition. The Fourier image of their product is the convolution

\*\*\*\*\*\*\*\* that is nonnegative everywhere

\*\*\*\*\*\*\*\* and serves as an analog of the Matsubara Green's function.

Proof. Note that the positive definite Toeplitz form \*\*\*\*\*\*\* has an absolutely integrable characteristic \*\*\*\*\*\* and combine this with the famous Bochner theorem.

An alternative proof via the second quantization is given in the author's monograph. Indeed, our Green function also can be defined through creation and annihilation operators. Resonance poles of the propagator \*\*\*\*\*\*\* may be construed as elementary excitations of our system.

**11. Quasiparticles.**

 Let us find the inverse effective masses of excitons with the help of residues.

Proposition. *Denote the zero of* \*\*\* *on the critical line by*

\*\*\*\*\*\*.

*One get the equality*

\*\*\*\*\*\*\*\*. *The role of the derivative at any zeta zero is now clarified.*

Proof. Here we look at the complex-valued function and integrate, bypassing the pole along the semicircle. The limit of our interest equals \*\*\*\*\*\*\*\*.

The integral over the rest part of the contour disappears as the small parameter tends towards zero. The integrand is in fact dominated by some integrable function, while \*\*\*\*\*\*\* for \*\*\*\*\*\*.

It remains to find the residue at the point

\*\*\*\*\*\*\*\*.

Let us multiply three Laurent series about it.

The first

\*\*\*\*\*\*\*\*\*\*

and the second

\*\*\*\*\*\*\*\* have no singularities, but the third

\*\*\*\*\*\*\*\*\* gives the pole of the second order.

So take the coefficient on the term \*\*\*\*\*\*\*\*\*\*

and find the desired residue

 \*\*\*\*\*\*\*\*. In order to find an integral around the pole, we need to multiply this value by \*\*\*\*.

Then multiply by \*\*\*\*, divide by \*\*\*\*\* and get the limit (bearing in mind that \*\*\*\*\* is a zeta zero).

**12. Mass gap.**

Let us say a few words about the physical sense of our theory. Looking at the Riemann Hypothesis as at spontaneous symmetry breaking,

we guess that the activation energy of excitons should be equal to zero as a manifestation of Goldstone's theorem.

But actually it is not so, inasmuch as we can establish the main result of our paper.

Theorem. *There is such a positive constant that*

\*\*\*\*\*\*\*\*\*\*.

*Consequently, all zeta zeros are simple, but the Riemann Hypothesis is false.*

Proof. Note that we are integrating against the Dirac delta-function

\*\*\*\*\*\*\*\*\*. Keep in mind the statement from the item \*\*\*\*\*.

Looking at the convolution and taking the limit, we can see with the help of the last statement from the item \*\*\*\*\* that the key inequality

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\* holds for all nontrivial zeta zeros on the critical line.

It is not difficult to prove the statement for the nontrivial zeros outside the critical line.

In this case, when \*\*\*\*\*\*, we must repeat our reasoning with respect to the Lagrangian \*\*\*\*\*\*\*\*.

In other words, there is a gap at the bottom of the energy spectrum

\*\*\*\*\*\*\*, and, by the way, we got the estimate of its size. To this end, we borrowed the numerical value of the coupling constant from the item \*\*\*. It is not without interest that our inequality is also valid for the trivial zeros of \*\*\*\*\*\*.

Simplicity of all zeros follows, whilst the disproof of the Riemann Hypothesis is ended by a modus tollens argument.

Mutatis mutandis, the same method is applied to the Dirichlet beta-function

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*.

It may be necessary to use the modified Lagrangian

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*

in order to prove the unconditional band gap

\*\*\*\*\*\*\*\*\*\*\*\*.

Nevertheless, under the Generalized Riemann Hypothesis

\*\*\*\*\*\*\*\*\*\*\*\*\*\*\*,

the conditional absence of the energy gap

\*\*\*\*\*\*\*\*\*\*\*\*\* turns out.

 After all, using the Probabilistic Number Theory, it is possible to derive the absence of the gap directly from the convergence of the Euler product.

That is why the corresponding Euler product

\*\*\*\*\*\*\* must diverge in the right half of the critical strip, and the Generalized Riemann Hypothesis should fail.

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