Against the Riemann Hypothesis

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Abstract

The author proposes a model quantum field theory with a mass gap — a positive constant low bound for absolute values of the Riemann zeta derivative at zeta zeros. This is evidence contrary to the Riemann Hypothesis.

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§ 1. Main result. The Riemann zeta - function is defined as the Dirichlet sum

.

In fact, has an analytic continuation over the whole complex plane with the simple pole as its only singularity.

It can be seen that trivially vanishes at the points

.

On the other hand, all nontrivial zeros occur in the strip

and *supposedly lie on the so - called critical line*

.

Under this Riemann Hypothesis [1], Ng conditionally showed [2] that

.

The author of the present paper suggests an innovative approach to the subject and proves the following unconditional statement.

1.1 THEOREM. *The Riemann Hypothesis is false because* .

We consider the values as the energy levels of elementary excitations [3] in a finely chosen oscillating system. It is to be hoped that our new vision may offer a solution to the most important problem of Mathematics [4].

§ 2. Positive definiteness. Before we touch upon the Riemann Hypothesis, we need some more background. Our concern here is the precise definition [5, 6] of reproducing kernels for the further study.

2.1 DEFINITION. A symmetric function

is called *positive definite* if the condition

holds for all finite sets of coefficients and arguments

.

This notion is defined in the same manner for bounded domains.

In order to construct new functions, we will apply the laws listed below:

- for each function both kernels

  ,

  should also be positive definite;

- in particular, functions of *product type*

  are always positive definite;

- multiplication by a positive number preserves positive definiteness;
- given a sequence of positive definite kernels

  $$, \quad , \dots \quad , \dots ,$$

  the sum of the series

  must be positive definite in its domain of convergence;
- futhermore, an integral function

  should be a positive definite kernel provided that the slices

  enjoy the same property.

From now on, we will use these rules without special announcement.

§ 3. Reproducing kernels. We begin with the following statement.

3.1 PROPOSITION. *Suppose that*

$$.$$

Then the function of two variables

is positive definite.

PROOF. First of all, let us put

$$.$$

Owing to the binomial formula

,

we get the decomposition

that converges on the bounded interval

.

One can easily see that the kernel

is a sum of a series of the product functions

with nonnegative coefficients. Thus, it is positive definite.

After multiplying by the product function, we come to the kernel

that is still positive definite.

Moreover, it appears possible to write the first

and the second

sums satisfying to the prerequisite condition

.

Termwise subtraction gives us the Taylor series

with nonnegative coefficients. By convention, .

Analysis similar to the above implies that the function

turns out to be positive definite.

Adding it twice to the previous one

,

we obtain another positive definite kernel. It remains to add the function

that is also positive definite. As a result, the total sum

is all the more positive definite. Without loss of generality, we change to . □

Deduce the next statement from the given exposition.

3.2 PROPOSITION. *Fix the value of the parameter*

.

Then the function

is positive definite.

PROOF. Let us produce a suitable change of variables

that keeps, as it always happens, the positive definiteness of the function

from indent **3.1**. Plug new arguments in the aforementioned form.

Firstly, we get the equality

and secondly, the identity

.

As a result, we obtain the positive definite function

.

The product function before the square brackets makes no influence on positive definiteness. So does the division by . □

We need the following auxiliary assertion.

3.3 PROPOSITION. *The function*

is positive definite under the same conditions.

PROOF. Using the Gamma distribution [7], we write an integral and give the analogous representation of its shift

.

Apparently, their difference

has the required property of positive definiteness.

Indeed, we see a continuous sum of the positive definite functions

.

Such terms of product type stand with the nonnegative weight

.

Division by the product function along with the positive external coefficient

retains the positive definiteness of the whole expression. □

We came to the important conclusion.

3.4 PROPOSITION. *The kernel*

is positive definite; here all parameters change within their former limits.

PROOF. We add the function

from item **3.2** to the function

from item **3.3**. Both of them are positive definite, and their sum is as well. □

Another observation concerns the double integral.

3.5 PROPOSITION. *The function*

is positive definite in the same circumstances.

PROOF. Accurate to the coefficient , the integrand has the form of the positive definite kernel from point **3.4**. We decompose it into an uniformly convergent series of product functions in view of Merser's theorem [8]. Double integration leads to another series of this type, and the sum is obviously positive definite. Besides, it can be considered as a correlation function of some integrated stochastic process [7], so its positive definiteness arises this way too. □

Here is the key outcome of this section.

3.6 PROPOSITION. *As usual, we get*

.

The function of two variables

is positive definite.

PROOF. Refine the double integral from indent **3.5**:

.

The multiplier before the square brackets does not affect the final conclusion. □

§ 4. Coupling constant. Now, before we go any further, let us establish a fact of great importance. The rest of the paper will throw light on its close links with Coulomb, or better to say, Yukawa's potential [9].

4.1 PROPOSITION. *There exists the constant C such that all kernels*

are positive definite; the domain of their definition remains the same.

PROOF. Double integration requires art, but its result can be routinely verified via Euler's theorem on homogeneous functions [10]. So that gives us

.

Or in another way,

.

Or even better,

.

Or last of all,

.

And in doing so, the double integral takes a form

.

Each kernel of our family is positive definite as an Hadamard product of the prefix

and the postfix

(accurate to the factor of product type).

The prefix is positive definite according to the statement **3.2**. To complete the proof we must show that the postfix is positive definite for sufficiently large and all possible , reducing the problem to the function of one variable

and its positive definiteness in the sense of Bochner [21].

It is positive definite as a limit of products of Cauchy distribution characteristic functions. Indeed, is nonvanishing and holomorphic in the complex plane with two cuts , so we can approximate it by nonvanishing rational functions with poles on these cuts. □

We can even estimate the numerical value of the absolute constant.

4.2 REMARK. *One can take* .

PROOF.

□

A crucial fact follows from these findings.

4.3 PROPOSITION. *The function of two variables*

is positive definite in its existence domain

with the given constant.

PROOF. It suffices to add together the function

from clause **4.1** and the function

from clause **3.6**.

Recall that the sum of two positive definite kernels has the same property. All rests on this principle. □

§ 5. Gibbs distribution. It is worth saying that any kernel is positive definite simultaneously with its double Laplace transform. Prove it by virtue of Merser's theorem and, conversely, on the grounds of the Müntz - Szász theorem [11].

The double Laplace transform of our next kernel is actually proportional to the positive definite function

from item **4.3**. Note that the positive proportionality coefficient preserves positive definiteness.

Let us turn to the explicit calculation of the double inverse Laplace transform. We prefer to promptly show the answer, and then we shall justify it.

5.1 PROPOSITION. *Under the standard conditions*

,

the kernel defined by the formula

is positive definite.

PROOF. So far as the multiplication by shifts the Laplace transform one unit on both variables and , we count on the operational relation

.

This formula seems, however, to be quite unobvious at the first glance.

Nevertheless, we instantly derive it from the more clear equation [12]

.

It is sufficient to find the joint density

and apply the standard rules of Operational Calculus [13]. □

When we place our oscillating system in the thermostat, is interpreted as a reciprocal temperature [10].

§ 6. Entropic factor. We introduce a very convenient notation

that shortens the record and highlights similarities with the entropy [10].

6.1 PROPOSITION. *The modified function*

is positive definite under ceteris paribus conditions.

PROOF. Our entropic factor is positive definite by itself, and we take an advantage of this observation. For example, another function

should be positive definite.

The kernel from point **5.1** is still positive definite after the division by the function of product type. Rewrite it in a slightly different way:

.

Redouble it and add the previous one

.

Through all necessary elementary transformations, we eventually obtain

as the positive definite kernel.

Since , adding the positive definite function

leads to the desired conclusion. □

§ 7. Microcanonical ensemble. Our next expression resembles the collision term of Boltzmann's kinetic equation [14] (Stosszahlansatz).

7.1 PROPOSITION. *Under all usual conditions, the function*

is positive definite.

PROOF. We use our favorite trick of multiplying by the product function

to deduce this proposition from the statement **6.1**. □

The passage from Statistical Mechanics to Euclidean Quantum Field Theory needs dimensionality reduction by Wick rotation, like always [15].

§ 8. Lagrangian. The entropic factor is homogeneous:

.

Here we are dealing with homogeneity of degree minus one [10].

8.1 PROPOSITION. *Each function from the family*

is positive definite.

PROOF. The change of variables

keeps the positive definiteness of the function from indent **7.1**.

We should recall that positive definiteness was defined in point **2.1** right down to the positive coefficient. One can neglect the positive number outside the brackets. □

The next statement receives a special name in view of its importance.

8.2 LEMMA. *The integral form with the fixed constant*

is positive definite in the domain

for each value of the main parameter

and is homogeneous of degree minus two.

PROOF. It is appropriate to introduce a notation for the Lagrangian [23]. Its homogeneity transpires from the simple change of variables:

.

This gauge invariance [16] reduces one degree of freedom.

As for positive definiteness, the integral sum of the positive definite slices from clause **8.1** has the same property. Remove the common entropic factor outside the integral sign. Its homogeneity was already mentioned. □

We are going to consider a positive definite Toeplitz form [17]. It depends only on the difference of variables.

8.3 COROLLARY. *We get a positive definite function* *generated by the characteristic*

when imposing the intrinsic constraint

between two previously independent arguments.

PROOF. The positive definiteness appears from the main lemma **8.2** and readily follows from the equality

,

which is not difficult to check. Owing to homogeneity, we write

.

It suffices to rearrange the variables

,

multiplying the product by the pair of exponents. □

§ 9. Normal modes. We bring our system to normal coordinates that behave just like the gas of uncoupled oscillators [22].

9.1 PROPOSITION. *The characteristic of the Lagrangian*

is decomposable into the double series.

PROOF. The fact under consideration is equivalent to the statement

.

In other words, the Lagrangian takes the form

.

In order to derive it from the very definition

,

let us integrate the decomposition

termwise. Helly's theorem justifies the integration process [21]. □

We dealt with elementary functions up to this point, but it is time to switch to the Riemann zeta.

9.2 PROPOSITION. *The formula*

defines the spectral density.

PROOF. Let us assume the translation principle of Harmonic Analysis to be known [19]. Apply this rule to the statement **9.1**. Shifts of a secant

to lags , with parallel divisions by act as multipliers

,

on the related Fourier transform. Termwise integration is justified via the Plancherel theorem [21]. □

Here is the first reference to the Riemann zeta. Because of the importance that has been placed upon the values inside the critical strip, we show first that $\zeta(s)$ admits an analytic continuation at least into this area.

Although it is useful to keep in mind different ways of analytic continuation as providing steps towards more general functions [1], we prefer to start with the most simple construction possible.

For such purpose, let us consider the Dirichlet eta - function defined as a series

that converges uniformly (but not absolutely) on compacts.

9.3 PROPOSITION. *We have expressed the spectral density*

through the Riemann zeta - function.

PROOF. The function

from point **9.2** is a well - known table integral [18].

On the other hand, the equality

holds. This calculation of the optical transfer function [19] is rather easy. □

The best effect is given by the use of Mellin transform [20], but the beauty of our method is due to the logic of appearance of .

9.4 PROPOSITION. *The characteristic of entropy*

reveals the Boltzmann factor of Gibbs ensemble.

PROOF. We get it from the definition

by the direct substitution. □

Now we are led to the Brownian oscillator with the viscosity δ.

9.5 PROPOSITION. *The formula*

defines the correspondent spectral selection filter.

PROOF. Guided by the previous statement **9.4**, we note that it is a question of the Fourier transform

.

We start from the table integral

and get

.

The sum of two integrals gives us the desired answer. □

§ 10. Green function. So when we come to the Källén - Lehmann spectral representation [15], let us recall the statement **8.3** and consider two characteristics from clauses **9.1** and **9.4**.

10.1 PROPOSITION. *The Fourier image of their product* [19]

is nonnegative everywhere and coincides with the convolution

that serves as an analog of Matsubara Green's function [22].

PROOF. Note that the absolutely integrable profile generates the positive definite Toeplitz form and apply the famous Bochner theorem [21]. □

An approach via Second Quantization was well developed [23] in the author's monograph. Indeed, our Green function can also be defined through creation and annihilation operators [9]. In such a way, resonance poles of the propagator may be construed as elementary excitations.

§ 11. Quasiparticles. We are going to compute the inverse effective masses of excitons with the help of residues [22].

11.1 PROPOSITION. *Denote any zero of on the critical line by*

.

The equality

clarifies the role of the derivative values at zeta zeros.

PROOF. Here we look at the complex - valued function and integrate, bypassing the pole along the semicircle. The limit of our interest equals

.

The integral over the rest part of the contour disappears as the small parameter tends towards zero: our integrand is in fact dominated by an integrable function, while for .

It remains to find the residue at the point .

So we have to multiply three Laurent series about it.

The first

and the second

have no singularities, but the third

gives the pole of second order. Hence, we must take the coefficient on the term

and calculate the desired residue

.

In order to find an integral around the pole, we need to multiply this value by . Then multiply by , divide by and get the limit (bearing in mind that is a zeta zero). □

§ 12. Mass gap. Let us discuss the physical sense of our theory. Treat the Riemann Hypothesis as spontaneous symmetry breaking and guess that the activation energy of excitons must equal zero as a manifestation of Goldstone's theorem [10]. Actually it is not so, inasmuch as we can establish the main result of our paper.

12.1 THEOREM. *There is such a positive constant that*

.

Consequently, all zeta zeros are simple and the Riemann Hypothesis is false.

PROOF. Note that we are integrating against the Dirac delta-function

.

Keep in mind the statement **10.1**.

Looking at the convolution and taking the limit, we see with the help of the last proposition **11.1** that the key inequality

holds for all nontrivial zeta zeros on the critical line.

It is not hard to prove this inequality for the nontrivial zeros outside the critical line. In such a case, when , we must repeat our reasoning with respect to the new Lagrangian

.

Therefore, there is a gap at the bottom of the energy spectrum

,

and, by the way, we got the estimate of its size. To this end, we borrowed the numerical value of the coupling constant from item **4.2**. It is not without interest that the last inequality is also valid for the trivial zeros of .

The proof of the Simplicity Hypothesis [1] follows. In light of [2], our disproof of the Riemann Hypothesis is ended by a Modus Tollens argument. □

Mutatis mutandis, apply the same method to the Dirichlet L-function [1]

.

In order to establish the unconditional band gap [24]

,

it may be necessary to employ the modified Lagrangian

.

Nevertheless, under the Generalized Riemann Hypothesis [1]

,

the conditional absence of the energy gap

turns out. We rely on the central limit theorem of Hejhal [25] after all.

In the spirit of Probabilistic Number Theory, it is natural to proceed directly from the Euler product

that must diverge in the right half of the critical strip. The failure of the Generalized Riemann Hypothesis puts some doubt on the BSD Conjecture as well [26].

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