Against the Riemann Hypothesis

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Abstract

The author proposes a model quantum field theory with a mass gap — a positive constant low bound for absolute values of the zeta derivative at zeta zeros. This is evidence contrary to the Riemann hypothesis.

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§ 1. Main Result

The zeta - function is defined as the Dirichlet sum

.

In fact, has an analytic continuation over the whole complex plane with the simple pole at as its only singularity.

It can be seen that trivially vanishes at the points

.

Meanwhile, all nontrivial zeta zeros occur in the strip

and *supposedly lie on the so - called critical line*

.

Under this hypothesis due to Riemann, Ng conditionally showed [20, Th. 1.3] that

.

The author of the present paper proves the following unconditional statement.

1.1 THEOREM. *The Riemann hypothesis is false because* .

An innovative approach to the subject suggests the proof: we consider the values as the energy levels of elementary excitations [24] in a finely chosen oscillating system. Our new vision may offer a solution to the most important problem of mathematics [6]. The word 'we' is hereinafter understood in the sense of pluralis auctoris.

§ 2. Positive Definiteness

Before we refer to the Riemann hypothesis [8], we need some more background. Our concern here is the precise definition of reproducing kernels [25] or correlation functions [22, p. 293] for a further study.

2.1 DEFINITION. A symmetric real function

is called *positive definite* if the condition

holds for all finite sets of coefficients and arguments

.

For bounded domains, this notion is defined in the same manner.

In order to construct new functions, we apply the laws listed below.

- For each function , both kernels

  must also be positive definite.

- In particular, functions of *product type*

  are always positive definite.

- Multiplication by a positive number preserves positive definiteness.
- Given a sequence of positive definite kernels

  ,

  we see that the sum of the series

  must be positive definite in its domain of convergence.
- Futhermore, an integral function

  must be positive definite provided that the slices

  enjoy the same property.

From now on, we will use these rules without special announcement.

§ 3. Reproducing Kernels

We begin with the following statement.

3.1 PROPOSITION. *Suppose that*

.

Then the function of two variables

(1)

is positive definite.

PROOF. First of all, let us put

.

Owing to the binomial formula

,

we have the decomposition

that converges on the bounded interval

.

One can easily see that the kernel

is a sum of a series of the product functions

with nonnegative coefficients and thus is positive definite [1, p. 261].

Multiplying by the product function, we come to another kernel

that is still positive definite.

Moreover, it appears possible to write the first

and the second

sums under the prerequisite condition

.

Termwise subtraction gives us the Taylor series

with nonnegative coefficients. By convention, .

Analysis similar to the above implies that the function

turns out to be positive definite.

Adding it twice to the previous one

,

we obtain a new positive definite kernel. It remains to add the function

that is also positive definite. The total sum

is all the more positive definite. Without loss of generality, we change to . □

Let us deduce the next statement from the given exposition.

3.2 Proposition. *Fix the value of the parameter*

.

Then the function

(2)

is positive definite.

PROOF. A suitable change of variables

keeps, as it always happens, the positive definiteness of the function (1)

from indent **3.1**. Plug new arguments in the aforementioned form.

First, we get the equality

and second, the identity

.

As a result, we obtain the positive definite function

.

The product before the square brackets has no influence on positive definiteness. Neither does the division by . □

We need the following auxiliary assertion.

3.3 PROPOSITION. *The function*

(3)

is positive definite under the same conditions.

PROOF. Using the gamma distribution [2, p. 25], we write an integral and give the analogous representation of its shift

.

Apparently, their difference

has the required property of positive definiteness.

Indeed, we see a continuous sum of the positive definite functions

.

Such terms of product type stand with the nonnegative weight

.

Division by the product function along with the positive external coefficient

retains the positive definiteness of the whole expression. □

We come to an important conclusion.

3.4 PROPOSITION. *The kernel*

(4)

is positive definite; here all parameters change within their former limits.

PROOF. We add the function (2)

from item **3.2** to the function (3)

from item **3.3**. Both of them are positive definite, and their sum is as well. □

Another observation concerns the double integral.

3.5 PROPOSITION. *The kernel*

(5)

is positive definite in the same circumstances.

PROOF. Accurate to the coefficient , the integrand has the form of the positive definite kernel (4) from point **3.4**. In view of Mercer's theorem [9, pp. 138-140], we decompose it into a uniformly convergent series of product functions. Double integration leads to another series of this type, and the sum is obviously positive definite. Also, a double integral of any correlation function can be considered as a correlation function of some integrated stochastic process [22, p. 289]. That gives an alternative proof of positive definiteness. □

Here is the key outcome of this section.

3.6 PROPOSITION. *Like previously, we have*

.

The function of two variables

(6)

is positive definite.

PROOF. Refine the formula (5) from indent **3.5**:

.

The multiplier outside the square brackets does not affect the final conclusion. □

§ 4. Coupling Constant

Now, before we go any further, let us establish a fact of great importance. The rest of the paper will throw light on close links with Coulomb, or better to say, Yukawa's potential [19].

4.1 Proposition. *There exists a constant such that all kernels*

(7)

are positive definite; their domain of definition remains the same.

Proof. Our statement amounts to one about the function

,

and the reader need not look far to explain its appearance.

According to the Nevanlinna - Pick theory [1, Th. 2.11], we have only to prove that the multiplication by in the double integral reproducing kernel space is a bounded operator with the norm less than .

The space consists of functions such that , where denotes the norm due to the reproducing kernel .

Nevanlinna and Pick's interpretation of item **3.2** yields the required bound

.

The δ - uniform constant upper estimate will be given below. □

Our approach uses the properties of the Cesàro operator [4].

4.2 REMARK. *One can take* .

PROOF. To bridge the gap in our knowledge about the Cesàro operator, we should shift attention to its definition

.

It is easy to note that the kernel

is positive definite, whenever it is positive definite for .

Thus [1, Th. 2.2], [25, p. 98], dilations

have norms , and the Cesàro operator is δ - uniformly bounded

.

Once we see matters in this light, the implications are twofold.

On the one hand,

□

A crucial fact follows from these findings.

4.3 PROPOSITION. *In its existence domain*

,

the function of two variables

(8)

is positive definite for the given constant.

PROOF. It suffices to add together the function (7)

from clause **4.1** and the function (6)

from clause **3.6**.

Recall that the sum of two positive definite kernels has the same property. All rests on this principle. □

§ 5. Gibbs Distribution

It is worth saying that kernels are positive definite simultaneously with their double Laplace transforms. Prove this by virtue of Mercer's theorem and, conversely, on the grounds of the double Widder - Post inversion formula [25, p. 75, Th. 2.5], [27, p. 277].

The double Laplace transform of our next kernel (9) is actually proportional to the positive definite function (8) from item **4.3**. The related scale factor

is positive, and (9) shall be, in its turn, positive definite.

Let us move to the explicit calculation of the double inverse Laplace transform of (8). We prefer to promptly show the answer, and then we will justify it.

5.1 PROPOSITION. *Under the standard conditions*

,

the kernel defined by the formula

(9)

is positive definite.

PROOF. So far as the multiplication by shifts the Laplace transform one unit on both variables and , we count on the operational relation

.

This formula seems, however, to be quite unobvious at the first glance.

Nevertheless, we instantly derive it from the clearer equation

.

See [10, pp. 359 - 360]. It is sufficient to find the joint density

and to apply the standard rules of operational calculus [10, p. 350, § 14]. □

When we place our oscillating system in a thermostat, is interpreted as a reciprocal temperature [7, p. 351] or as an imaginary time [28, p. 289].

§ 6. Entropy Factor

We introduce a convenient notation

that shortens the record and highlights similarities with the entropy [7, p. 329].

6.1 PROPOSITION. *The modified function*

is positive definite under ceteris paribus conditions.

PROOF. Our entropy factor is positive definite by itself, and we take advantage of this observation. For example, another function

must be positive definite.

The kernel (9) from point **5.1** is still positive definite after the division by the function of product type. Rewrite it in a slightly different way:

Redouble it and add the previous one

Through all necessary elementary transformations, we eventually obtain

as the positive definite kernel.

Since , adding the positive definite function

leads to the desired conclusion. □

§ 7. Microcanonical Ensemble

Our next expression resembles the collision term (*Stosszahlansatz*) of Boltzmann's kinetic equation [16].

7.1 PROPOSITION. *Under all usual conditions, the function*

$$ \tag{10}$$

is positive definite.

PROOF. We use our favorite technique of multiplying by the product function

to deduce this proposition from statement **6.1**. □

§ 8. Lagrangian

The passage from statistical mechanics to euclidean quantum field theory needs dimensionality reduction by Wick rotation, like always [28, p. 287].

The entropy factor is homogeneous:

.

Here we deal with homogeneity of degree minus one [7, p. 28].

8.1 PROPOSITION. *Each function from the family*

$$ \tag{11}$$

is positive definite.

PROOF. The change of variables

keeps the positive definiteness of the function (10) from indent **7.1**.

We should recall (see point **2.1**) that positive definiteness was defined right down to a positive coefficient. One can neglect the positive number outside the brackets. □

The next statement receives a special name in view of its importance.

8.2 LEMMA. *For each value of the main parameter*

,

the integral form with the fixed constant

is positive definite in the domain

and is homogeneous of degree minus two.

PROOF. It is appropriate to introduce a notation for the Lagrangian [12]. Homogeneity transpires from the simple change of variables:

.

This gauge invariance [26] reduces one degree of freedom.

As for positive definiteness, the integral of the positive definite slices (11) from clause **8.1** has the same property. Remove the common entropy factor outside the integral sign. Its homogeneity was already mentioned. □

Now we consider a positive definite Toeplitz form [14]. It depends only on the difference of variables, like in stationary systems whose features do not vary with time (or in space).

8.3 Corollary. *We face a positive definite function*

generated by the characteristic

(12)

when imposing the intrinsic constraint

between two previously independent arguments.

Proof. The positive definiteness appears from lemma **8.2** and readily follows from the equality

,

which is not difficult to check. Owing to homogeneity, we write

, .

It is sufficient to rearrange the variables

,

multiplying the product by the pair of exponentials. □

§ 9. Normal Modes

We bring our system to normal coordinates that behave just like the gas of uncoupled oscillators [15, pp. 18 - 22].

9.1 Proposition. *The characteristic of the Lagrangian*

is decomposable into the double series.

PROOF. The fact under consideration is equivalent to the identity

.

In other words, the Lagrangian takes the form

.

In order to derive it from the very definition

,

let us integrate the decomposition

termwise. Helly's theorem justifies the integration process [18, p. 44]. □

Up to this point, we dealt with elementary functions. It is time to switch to the Riemann zeta.

9.2 PROPOSITION. *The formula*

defines the spectral amplitude.

PROOF. Termwise integration is justified via the Plancherel theorem [18, p. 76]. Let us assume the translation principle of harmonic analysis [23, Ch. 3, § 1] to be known. Combine this rule with statement **9.1**. Shifts of a secant to lags with parallel divisions by act as multipliers

on the related Fourier transform. □

Here is the first reference to the Riemann zeta. Because of the importance placed upon its values inside the critical strip, we show first that admits an analytic continuation at least into this area.

Although it is useful to keep in mind different ways of analytic continuation as providing steps toward more general functions [8, p. 347], we prefer to start with the most simple construction possible.

For such purpose, let us consider the Dirichlet eta-function defined as a series

that converges uniformly on compacts.

9.3 PROPOSITION. *We express the spectral amplitude*

(13)

through the Riemann zeta.

PROOF. The function

from point **9.2** is a well-known table integral [11, № 861.62, № 850.3].

On the other hand, the equality

holds. This calculation of the optical transfer function [23, Ch. 2] is direct. □

One can avoid termwise integration by the use of the Mellin transform [21], but the beauty of our method is due to the logic of the appearance of .

9.4 PROPOSITION. *The entropy characteristic*

reveals the Boltzmann factor of the canonical ensemble.

PROOF. We obtain it from the definition

by the straightforward substitution. □

So we are led to a Brownian oscillator [22, p. 326] with the viscosity .

9.5 PROPOSITION. *The formula*

(14)

defines the correspondent spectral selection filter.

PROOF. Guided by statements **9.2** and **9.4**, we note that it is a question of the Fourier transform

.

We start from the integral

and get the first moment

.

The sum of two integrals gives us the desired answer. □

§ 10. Green Function

Now when we come to the Källén - Lehmann spectral representation [3, p. 558], let us recall the statement from clause **8.3** and consider two signals (13), (14) from clauses **9.3** and **9.5**.

10.1 PROPOSITION. *The Fourier image of the product* (12)

(15)

is nonnegative everywhere and coincides with the convolution [23, Ch. 3, § 2]

that serves as an analog of the Matsubara Green's function [28].

PROOF. Since the absolutely integrable characteristic generates the positive definite Toeplitz form , we apply the famous Bochner theorem [18, p. 71, Th. 4.2.2]. □

§ 11. Quasiparticles

An approach via second quantization was well developed [12] in the author's monograph. Our Green function can also be defined through creation and annihilation operators [19]. In such a way, resonance poles of the propagator may be construed as elementary excitations.

11.1 PROPOSITION. *Denote by*

any zero of *on the critical line. The equality*

(16)

clarifies the role of the derivative at zeta zeros.

PROOF. Let us compute the inverse effective masses of excitons with the help of residues [15, p. 270]. Here we look at the complex - valued function and integrate, bypassing the pole along the semicircle. The limit (16) equals

.

The integral over the rest part of the contour disappears as the small parameter tends toward zero: our integrand is in fact dominated by an integrable function, while for .

It remains to find the residue at the point .

So we multiply three Laurent series about it.

The first

and the second

have no singularities, but the third

gives the pole of second order. Hence, we take the coefficient on the term

and calculate the desired residue

.

In order to find an integral around the pole, we need to multiply by . Then multiply by , divide by and get the limit, bearing in mind that is a zeta zero. □

§ 12. Mass Gap

Let us guess the physical meaning of our theory. In strict accordance with Goldstone's theorem on spontaneous symmetry breaking [7, p. 462], the Riemann hypothesis would oblige the activation energy of excitons to be equal to zero. By contraposition, the main result of the present paper contradicts the Riemann conjecture.

12.1 Theorem. *There is a positive constant such that*

. (17)

Consequently, all zeta zeros are simple, and the Riemann hypothesis is false.

Proof. Note that we integrate against the Dirac delta-function

.

Keep in mind the statement from point **10.1**.

The convolution structure of (15) allows us to pass to the limit with the help of the formula (16) from proposition **11.1**. We see that the key inequality

holds for all nontrivial zeta zeros on the critical line.

This is not hard to prove for the nontrivial zeros

outside the critical line. In such a case, we repeat our reasoning with respect to a new Lagrangian

.

Therefore, there is a gap at the bottom of the energy spectrum, and, by the way, we get the estimate of its size.

To this end, we use the value of the coupling constant from item **4.2**. It is not without interest that the lower bound

is also valid for the trivial zeros of .

The reader is left with the task of determining the numerical value of the absolute constant . We suggest an answer that is merely a rough estimate, but finding an optimal solution is beyond the scope of our study.

The simplicity hypothesis [8, p. 353] immediately follows from (17). In light of [20, Th. 1.3], our disproof of the Riemann hypothesis is ended by a modus tollens argument. □

Mutatis mutandis, the same method works for the Dirichlet L - function [8]

.

Namely, the author employed the modified Lagrangian

to establish the unconditional *band gap* [5]

.

Nevertheless, the conditional absence of this gap

stems from the generalized Riemann hypothesis [8]

.

We rely on the central limit theorem of Hejhal [17] after all.

The only way out is to abandon the generalized Riemann hypothesis.

In the spirit of probabilistic number theory, it is natural to proceed directly from the Euler product

that now *must diverge in the right half of the critical strip.*

The last circumstance casts grave doubt on the Birch and Swinnerton - Dyer conjecture [13].

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