Physics of the Riemann Zeros: the Low Bound for the Zeta Derivative via Quantum Field Theory

Andrey Vlad. Egorov, Ph.D

—A product of the specific Lagrangian and the entropy factor is defined. Its positive definiteness is stated for the proper coupling constant. The passage from statistical mechanics to quantum field theory is performed by Wick rotation. The Green function (a convolution of the spectral amplitude and the propagator) is positive. Masses of quasiparticles are computed as residues. The role of the zeta derivative at zeta zeros is then highlighted, and the correspondent low bound is obtained.

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I. Statistical Mechanics

For all positive values

$$0 < v, w < \infty,$$

a model Lagrangian of the oscillating system

$$\mathcal{L}_\delta(v, w) \overset{\text{def}}{=\!=}$$

$$\overset{\text{def}}{=\!=} \int\limits_0^\infty \left[\frac{C^2}{\delta^2} \cdot \frac{e^{-(v+w)u}}{(1+e^{-vu})\,(1+e^{-wu})} - e^{-(v+w)u} \right] du$$

is proposed.

When this system is placed in a thermostat, an entropy factor

$$\mathcal{E}_\delta(v, w) \overset{\text{def}}{=\!=}$$

$$\overset{\text{def}}{=\!=} -\ln \left\{ \frac{\min{(v, w)}^{2\delta}}{e(vw)^\delta} \right\} \frac{\min{(v, w)}^{2\delta}}{(vw)^{1/2+\delta}}$$

comes into play.

The product

$$\{\mathcal{L}_\delta \cdot \mathcal{E}_\delta\}(v, w) \overset{\text{def}}{=\!=} \mathcal{L}_\delta(v, w) \cdot \mathcal{E}_\delta(v, w)$$

resembles the collision term of Boltzmann's kinetic equation [5]. The free small parameter

$$0 < \delta < \frac{1}{2}$$

can be interpreted as a reciprocal temperature or as an imaginary time [9].

e-mail: egorovmillenium@gmail.com
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Lemma. *For a sufficiently large coupling constant* C, *the positive definiteness property*

$$\sum_{j,\,k} \{\mathcal{L}_\delta \cdot \mathcal{E}_\delta\}(v_j, v_k) f_j f_k \geq 0$$

is valid with all finite sets of numbers

$$0 < v_j < \infty, \ -\infty < f_j < \infty$$

and all δ.

On the microcanonical level, it is enough to prove the positive definiteness [1] of the Nevanlinna-Pick kernels

$$\left[\frac{C^2}{\delta^2} \cdot \frac{e^{-v} e^{-w}}{(1+e^{-v})\,(1+e^{-w})} - e^{-v} e^{-w} \right] \mathcal{E}_\delta(v, w),$$

or, equivalently, of the elementary functions

$$\left[C^2 - \delta^2 \left(1+e^{-v}\right) \left(1+e^{-w}\right) \right] \mathcal{E}_\delta(v, w).$$

This was done by means [7] of the reproducing kernel Hilbert space theory.

II. Dirichlet Eta

Before we start out discussing the zeros of $\zeta(s)$, we need some background. Our concern here is the precise definition of the main object for the further study. Prior to the discussion on this essential object, we now introduce some notation which seems convenient for the present article.

It should be fixed that the Riemann zeta-function is defined as the Dirichlet sum

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}, \ s \overset{\text{def}}{=\!=} \sigma + it,$$

where the series converges absolutely in the half-plane $\sigma > 1$. The significance of this function grows from the fact that the sum of the series may possess a holomorphic continuation outside its region of convergence.

Because of the importance [3] placed upon its zeros inside the critical strip $0 < \sigma < 1$, one must show that $\zeta(s)$ admits an analytic continuation at least into this area. Here is the first reference to the Dirichlet eta-function

$$\eta(s) \overset{\text{def}}{=\!=} \left(1 - \frac{2}{2^s}\right) \zeta(s) = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n^s},$$

$$\sigma > 0$$

that serves for such purpose.

III. Quantum Field Theory

The passage from statistical mechanics to euclidean quantum field theory needs dimensionality reduction [9].

Consider a positive definite Toeplitz form [4]

$$\{L_\delta \cdot E_\delta\}(x-y) \stackrel{\text{def}}{=\!=} \{\mathcal{L}_\delta \cdot \mathcal{E}_\delta\}\left(e^{(x-y)/2}, e^{-(x-y)/2}\right) =$$

$$= e^x e^y \{\mathcal{L}_\delta \cdot \mathcal{E}_\delta\}(e^x, e^y).$$

It depends only on the difference of variables

$$-\infty < x, y < \infty,$$

like in stationary systems whose features do not vary with time (or in space).

Bring our system to normal coordinates that behave just like the gas of uncoupled oscillators.

Now define the Green function with the help of the Fourier transform

$$\{\widehat{L_\delta \cdot E_\delta}\}(t) \stackrel{\text{def}}{=\!=} \int\limits_{-\infty}^{\infty} e^{-ixt}\{L_\delta \cdot E_\delta\}(x)\,dx \geq 0.$$

It is nonnegative everywhere along

$$-\infty < t < \infty$$

due to the Bochner theorem [8].

Our Green function can be calculated as a convolution [6]

$$\{\widehat{L_\delta \cdot E_\delta}\}(t) = \frac{\widehat{L}_\delta \star \widehat{E}_\delta(t)}{2\pi} =$$

$$= \frac{2}{\pi}\int\limits_{-\infty}^{\infty}\left|\Gamma\left(\frac{1}{2}+i\tau\right)\right|^2 \left\{\frac{C^2}{\delta^2}\left|\eta\left(\frac{1}{2}+i\tau\right)\right|^2 - 1\right\} \times$$

$$\times \frac{\delta^3\,d\tau}{\left[\delta^2 + (\tau - t)^2\right]^2}$$

and even expressed through creation and annihilation operators. Such approach via second quantization was also developed.

In a certain way, resonance poles of the propagator

$$\widehat{E}_\delta(t) = \frac{4\delta^3}{(\delta^2 + t^2)^2}$$

may be construed as elementary excitations or quasiparticles. The inverse effective masses of excitons must be computed with the help of residues. Denote by

$$\varrho \stackrel{\text{def}}{=\!=} \frac{1}{2} + i\gamma$$

any zero of $\zeta(s)$ on the critical line. The equality

$$\lim_{\delta\downarrow 0}\frac{2}{\pi\delta^2}\int\limits_{-\infty}^{\infty}\left|\{\Gamma\cdot\eta\}\left(\frac{1}{2}+i\tau\right)\right|^2 \frac{\delta^3\,d\tau}{\left[\delta^2 + (\tau-\gamma)^2\right]^2} =$$

$$= \left|\{\Gamma\cdot\eta\}'(\varrho)\right|^2$$

clarifies the role of the derivative at zeta zeros.

The key inequality

$$\lim_{\delta\downarrow 0}\{\widehat{L_\delta \cdot E_\delta}\}(\gamma) =$$

$$= |\Gamma(\varrho)|^2\left\{C^2\left|1-2^{1-\varrho}\right|^2|\zeta'(\varrho)|^2 - 1\right\} \geq 0$$

holds for all nontrivial zeta zeros on the critical line.

Theorem. *The lower bound*

$$|\zeta'(\varrho)| \geq \frac{1}{3\cdot C} = c > 0$$

is a gap in the activation energy spectrum.

Our efforts have principally been directed toward getting information about gaps between zeta zeros.

IV. Gaps between Zeros

Let us apply the trick which is worth knowing because it turns up in a variety of contexts. For the consecutive pair γ_n and γ_{n+1}, one can write

$$0 = \int\limits_{\gamma_n}^{\gamma_{n+1}} \zeta'\left(\frac{1}{2}+it\right)dt =$$

$$= -\int\limits_{\gamma_n}^{\gamma_{n+1}} \zeta'\left(\frac{1}{2}+it\right)d(\gamma_{n+1}-t) =$$

$$= (\gamma_{n+1}-\gamma_n)\,\zeta'\left(\frac{1}{2}+i\gamma_n\right) +$$

$$+\, i\int\limits_{\gamma_n}^{\gamma_{n+1}} (\gamma_{n+1}-t)\,\zeta''\left(\frac{1}{2}+it\right)dt$$

and conclude that

$$\frac{-\,i}{\zeta'(1/2+i\gamma_n)}\int\limits_{\gamma_n}^{\gamma_{n+1}} (\gamma_{n+1}-t)\,\zeta''\left(\frac{1}{2}+it\right)dt =$$

$$= \gamma_{n+1} - \gamma_n.$$

In fact, this is a realistic way of bounding the gaps from below. One can make use of the inequality

$$\left|\zeta'\left(\frac{1}{2}+i\gamma_n\right)\right| > c > 0$$

and get for $\gamma_n \to \infty$ an estimate

$$\gamma_{n+1} - \gamma_n =$$

$$= O\left\{(\gamma_{n+1}-\gamma_n)^2 \max_{\gamma_n < t < \gamma_{n+1}}\left|\zeta''\left(\frac{1}{2}+it\right)\right|\right\}.$$

In order to find low bounds

$$\left\{\max_{\gamma_n < t < \gamma_{n+1}}\left|\zeta''\left(\frac{1}{2}+it\right)\right|\right\}^{-1} \ll \gamma_{n+1} - \gamma_n$$

for all natural n, let us say a few words about the growth of the zeta-function along the critical line.

V. Separation Condition

The order of growth

$$\mu(\sigma) = \limsup_{t \to \infty} \frac{\ln|\zeta(\sigma + it)|}{\ln t}$$

should not be confused with the Möbius function. Due to the functional equation, it is sufficient to consider only the values $\sigma \geq 1/2$. Moreover, $\mu(\sigma)$ is a continuous, nonincreasing and convex downwards function.

Our state of knowledge is not satisfactory, and the problem of determining the precise order of growth is yet unsolved, so we can focus on the central value

$$\mu\left(\frac{1}{2}\right).$$

We have an upper bound for this value that was improved many times:

- Hardy and Littlewood gave the upper bound $1/6$;
- Walfisz — $163/988$;
- Titchmarch — $27/164$;
- Phillips — $229/1392$;
- Titchmarch — $15/92$;
- Min — $19/116$;
- Haneke and Jin-run — $6/37$;
- Kolesnik — $35/216$.

Recently, Bourgain [2] gave the estimate

$$13/84 = 0.154\ldots,$$

which broke the 0.16-barrier.

His result can be transferred to the second derivative

$$\left|\zeta''\left(\frac{1}{2} + it\right)\right| = O\left(t^{0.16}\right), \quad t \to \infty$$

through the Cauchy formula.

Corollary. *Compare this estimate with the last inequality and see that the separation condition*

$$\gamma_n^{-0.16} \ll \gamma_{n+1} - \gamma_n$$

holds for $n \to \infty$.

The Lindelöf hypothesis claims that

$$\mu(\sigma) = \begin{cases} 0, & \sigma \geq 1/2 \\ 1/2 - \sigma, & \sigma < 1/2, \end{cases}$$

but the exact value of $\mu(\sigma)$ remains unknown.

VI. Perspectives

According to Brillouin and Parodi, the band gap effect is closedly connected with the propagation of waves in the periodic structures. Indeed, the Lindelöf hypothesis combined with the theorem above leads to the separation condition of such kind. Goldston stressed that the existence of a Siegel zero would force all the gaps between the consecutive zeros in a certain large range to never be closer than half times average spacing, and also have more unlikely but still possible properties. He underlined that all existing methods exhibit the presumed barrier at half times average spacing for small gaps, and we do not know any infinite sequences of non-zero gaps much shorter than the average spacing.

Thus, the very strange but stubborn *Alternative hypothesis* suggests that, asymptotically, the normalized gaps

$$\frac{\gamma_n \ln \gamma_n - \gamma_{n+1} \ln \gamma_{n+1}}{2\pi}$$

between the consecutive zeros ϱ_n and ϱ_{n+1} are all nonzero integers or half-integers.

It is known that pursuing consequences of this conjecture will shed light on two mysterious problems — the problem of existence of infinite number of twin primes and the problem of the existence of infinite number of Siegel zeros.

VII. Generalizations

Is not hard to generalize our theorem for the zeros

$$\varrho = \frac{1}{2} + \varepsilon + i\gamma$$

outside the critical line.

Namely, one can repeat the same reasoning with respect to the modified Lagrangian

$$\int_0^\infty \left[\frac{C^2}{\delta^2} \cdot \frac{e^{-(v+w)u}}{(1+e^{-vu})(1+e^{-wu})} - e^{-(v+w)u}\right] \frac{du}{u^\varepsilon}$$

and obtain the same gap at the bottom of the energy spectrum. Mutatis mutandis, the method also works for the beta-function

$$\beta(s) \stackrel{\text{def}}{=\!=} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}, \ \sigma > 0.$$

In such a case, the author employed the model

$$\int_0^\infty \left[\frac{C^2}{\delta^2} \cdot \frac{e^{-(v+w)u}}{(1+e^{-2vu})(1+e^{-2wu})} - e^{-(v+w)u}\right] \frac{du}{u^\varepsilon}$$

to establish the band gap.

VIII. Formulas

The entropy characteristic

$$E_\delta(x) \stackrel{\text{def}}{=\!=} \mathcal{E}_\delta\left(e^{x/2}, e^{-x/2}\right) = (1 + \delta|x|)e^{-\delta|x|}$$

reveals the Boltzmann factor of the canonical ensemble.

Obtain it from the definition

$$\mathcal{E}_\delta(v, w) \stackrel{\text{def}}{=\!=} -\ln\left\{\frac{\min(v,w)^{2\delta}}{e(vw)^\delta}\right\} \frac{\min(v,w)^{2\delta}}{(vw)^{1/2+\delta}}$$

by the straightforward substitution. The formula

$$\widehat{E}_\delta(t) = \frac{4\delta^3}{(\delta^2 + t^2)^2}$$

defines the correspondent spectral selection filter.

So we are led to a Brownian oscillator with the viscosity δ.

Note that it is a question of the Fourier transform

$$\widehat{E}_\delta(t) \stackrel{\text{def}}{=\!=} \int_{-\infty}^{\infty} e^{-ixt} E_\delta(x)\, dx =$$

$$= \int_{-\infty}^{\infty} e^{-ixt}\left(1 + \delta|x|\right) e^{-\delta|x|}\, dx.$$

Start from the integral

$$\int_{-\infty}^{\infty} e^{-ixt} e^{-\delta|x|}\, dx = \frac{1}{\delta - it} + \frac{1}{\delta + it} = \frac{2\delta}{\delta^2 + t^2} =$$

$$= \frac{2\delta^3}{(\delta^2+t^2)^2} + \frac{2\delta t^2}{(\delta^2+t^2)^2}$$

and get the first moment

$$\int_{-\infty}^{\infty} e^{-ixt}\, \delta|x|\, e^{-\delta|x|}\, dx = -\,\delta\, \frac{d}{d\delta} \int_{-\infty}^{\infty} e^{-ixt} e^{-\delta|x|}\, dx =$$

$$= \frac{2\delta^3}{(\delta^2+t^2)^2} - \frac{2\delta t^2}{(\delta^2+t^2)^2}.$$

The sum of two integrals gives us the desired answer.

The Lagrangian takes the form

$$\mathcal{L}_\delta(v, w) \stackrel{\text{def}}{=\!=}$$

$$\stackrel{\text{def}}{=\!=} \int_0^{\infty} \left[\frac{C^2}{\delta^2} \cdot \frac{e^{-(v+w)u}}{(1+e^{-vu})\,(1+e^{-wu})} - e^{-(v+w)u}\right] du =$$

$$= \frac{C^2}{\delta^2} \sum_{n,\, m\, =\, 1}^{\infty} \frac{(-1)^{n+m}}{nv + mw} - \frac{1}{v+w}.$$

Derive it from the very definition integrating the series

$$\frac{e^{-vu} e^{-wu}}{(1+e^{-vu})\,(1+e^{-wu})} =$$

$$= \sum_{n,\, m\, =\, 1}^{\infty} (-1)^{n+m} \exp\left\{-\left(nv + mw\right) u\right\}$$

termwise.

The characteristic of the Lagrangian

$$L_\delta(x) \stackrel{\text{def}}{=\!=} \mathcal{L}_\delta\left(e^{\,x/2}, e^{-x/2}\right) =$$

$$= \frac{C^2}{\delta^2} \sum_{n,\, m\, =\, 1}^{\infty} \frac{(-1)^{n+m}}{n e^{x/2} + m e^{-x/2}} - \frac{1}{e^{x/2} + e^{-x/2}} =$$

$$= \frac{C^2}{\delta^2} \sum_{n,\, m\, =\, 1}^{\infty} \frac{(-1)^{n+m}}{2\sqrt{nm}\, \cosh\left(\frac{x + \ln n - \ln m}{2}\right)} -$$

$$- \frac{1}{2\cosh\left(\frac{x}{2}\right)}$$

is decomposable into the double series.

Let us assume the translation principle of harmonic analysis to be known. Shifts of a secant

$$\frac{1}{2\cosh\left(\frac{x}{2}\right)}$$

to various independent lags

$$\ln n, -\ln m$$

with parallel divisions by numbers

$$\sqrt{nm}$$

act on the related Fourier transform as multipliers

$$\frac{n^{it} \cdot m^{-it}}{\sqrt{n} \cdot \sqrt{m}}.$$

Therefore, the final formula

$$\widehat{L}_\delta(t) \stackrel{\text{def}}{=\!=} \int_{-\infty}^{\infty} e^{-ixt} L_\delta(x)\, dx =$$

$$= \int_{-\infty}^{\infty} \frac{e^{-ixt}\, dx}{2\cosh\left(\frac{x}{2}\right)} \left\{\frac{C^2}{\delta^2} \sum_{n,\, m\, =\, 1}^{\infty} \frac{(-1)^{n+m} \cdot n^{it} \cdot m^{-it}}{\sqrt{n} \cdot \sqrt{m}} - 1\right\}$$

defines the spectral amplitude.

The function

$$\int_{-\infty}^{\infty} \frac{e^{-ixt}\, dx}{2\cosh\left(\frac{x}{2}\right)} = \frac{\pi}{\cosh(\pi t)} = \left|\Gamma\left(\frac{1}{2} + it\right)\right|^2$$

is a well-known table integral. On the other hand, the equality

$$\sum_{n,\, m\, =\, 1}^{\infty} \frac{(-1)^{n+m} \cdot n^{it} \cdot m^{-it}}{\sqrt{n} \cdot \sqrt{m}} = \left|\eta\left(\frac{1}{2} + it\right)\right|^2$$

holds.

Now when it comes to the Källén - Lehmann representation, the spectral amplitude

$$\widehat{L}_\delta(t) = \left|\Gamma\left(\frac{1}{2} + it\right)\right|^2 \left\{\frac{C^2}{\delta^2} \left|\eta\left(\frac{1}{2} + it\right)\right|^2 - 1\right\}$$

is expressed through the Riemann zeta.

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